ON CHARACTERIZATION OF INTEGRABLE SESQUILINEAR FORMS

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ABSTRACT. We give necessary and sufficient condition for a sesquilinear form to be integrable with respect to a faithful normal state on a von Neumann algebra.

The fundamental solution to the problem of constructing a noncommutative $L_1(\varphi)$-space associated with a faithful normal semifinite weight φ on a von Neumann algebra $\mathcal{M}$ was obtained in 1972–78. This space was realized as a space of "integrable" sesquilinear forms defined on a "lineal of weight" and "affiliated" with $\mathcal{M}$. In the next years this approach was thoroughly developed (see the survey [7] and the monograph [9]). For the other approaches to the integration with respect to weights and states we refer the reader to the surveys [7], [4] and the recent paper [3].

It is well known that a bounded linear operator on a Hilbert space is nuclear if and only if it has finite matrix trace (see for instance [2, Theorem III.8.1]). In the present paper we examine a problem whether certain analogue of that assertion holds for integrable sesquilinear forms.

In what follows, H is a Hilbert space with the scalar product denoted by $\langle\cdot,\cdot\rangle$.

Let φ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$ of operators on H (see, e. g., [6]), $\mathfrak{m}_\varphi^+ = \{x \in \mathcal{M}^+ \colon \varphi(x) < +\infty\}$, $\mathfrak{m}_\varphi^{\mathrm{sa}} = \mathfrak{m}_\varphi^+ - \mathfrak{m}_\varphi^+$. It is well known that the formula

$$\|x\|_\varphi \equiv \inf\{\varphi(x_1+x_2)\colon x = x_1 - x_2;\ x_1, x_2 \in \mathfrak{m}_\varphi^+\}$$

determines a norm $\|\cdot\|_\varphi$ on $\mathfrak{m}_\varphi^{\mathrm{sa}}$. By $L_1(\varphi)^{\mathrm{sa}}$ we will denote the corresponding completion of $\mathfrak{m}_\varphi^{\mathrm{sa}}$.

The linear subspace of H

$$D_\varphi \equiv \{f \in H \colon \exists \lambda > 0\ \forall x \in \mathcal{M}^+\ (\langle xf, f\rangle \leqslant \varphi(x))\}$$

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was introduced and called *the lineal of weight* in [8]. Clearly, if φ is represented in the form

$$\varphi = \sum_{i \in I} \langle \cdot f_i, f_i \rangle, \quad f_i \in H, \tag{1}$$

then $f_i \in D_\varphi$ $(i \in I)$.

The real Banach space $L_1(\varphi)^{\mathrm{sa}}$ can be realized by hermitian sesquilinear forms defined on D_φ. Namely, if $\widetilde{x} \in L_1(\varphi)^{\mathrm{sa}}$ and (x_n) is a Cauchy sequence in the normed space $(\mathfrak{m}_\varphi^{\mathrm{sa}}, \|\cdot\|_\varphi)$, which determines the element $\widetilde{x}$ of the completion, then the formula

$$a_{\widetilde{x}}(f, g) = \lim_n \langle x_n f, g \rangle, \qquad f, g \in D_\varphi,$$

correctly defines a hermitian sesquilinear form $a_{\widetilde{x}}$. The sequence (x_n) is called *defining for* $a_{\widetilde{x}}$. Also, since $|\varphi(x)| \le \|x\|_\varphi$ for any $x \in \mathfrak{m}_\varphi^{\mathrm{sa}}$, the formula

$$\varphi(a_{\widetilde{x}}) = \lim_n \varphi(x_n)$$

correctly defines the value $\varphi(a_{\widetilde{x}})$ which is called *the integral* (or *the expectation*) of the sesquilinear form $a_{\widetilde{x}}$ with respect to φ. Accordingly, such sesquilinear forms are called *integrable*. Moreover, the main result of [8] (Theorem 2) says that the map $\widetilde{x} \mapsto a_{\widetilde{x}}$ $(\widetilde{x} \in L_1(\varphi)^{\mathrm{sa}})$ is injective (see also [9, Theorem 16.7], [7, Theorem 1]). Thus, $L_1(\varphi)^{\mathrm{sa}}$ is meaningfully described as a real Banach space of integrable sesquilinear forms. The cone $L_1(\varphi)^+$ of integrable positive sesquilinear forms induces a natural order structure in $L_1(\varphi)^{\mathrm{sa}}$. The space $L_1(\varphi)$ is defined as a certain complexification of $L_1(\varphi)^{\mathrm{sa}}$ [9, 16.11], [7, 1.5], and the notion of the integral is extended to sesquilinear forms in $L_1(\varphi)$. The following proposition gives an "explicit" form of such integral.

Proposition 1 ([9, Proposition 17.11])**.** *Let*

$$\varphi = \sum_{i \in I} \langle \cdot f_i, f_i \rangle, \quad f_i \in H, \tag{1}$$

be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$ *and* $a \in L_1(\varphi)$*. Then*

$$\varphi(a) = \sum_{i \in I} a(f_i, f_i), \tag{2}$$

where the series in (2) *converges absolutely and its sum does not depend on the choice of representation of* φ *in the form* (1).

In [9, page 166], the following problem was posed: does the converse to Proposition 1 hold? The theorem below gives an affirmative answer to the question in the special case of normal states.

Theorem 2. *Let φ be a faithful normal state on a von Neumann algebra $\mathcal{M}$. For a sesquilinear form a defined on D_φ, the following conditions are equivalent:*

(i) $a \in L_1(\varphi)$,

(ii) *for any representation $\varphi = \sum_{i\in I} \langle \cdot f_i, f_i\rangle$, the series $\sum_{i\in I} a(f_i, f_i)$ converges absolutely and the sum does not depend on the representation of φ.*

Proof. By virtue of Proposition 1, it suffices to prove (ii) $\Longrightarrow$ (i). Moreover, it is clear that we can restrict ourselves to the case when a is hermitian.

So, let φ be a faithful normal state on $\mathcal{M}$ and a hermitian sesquilinear form a on D_φ satisfy (ii).

Denote by Y the Banach space of hermitian σ-weakly continuous functionals ψ on $\mathcal{M}$ such that $-\lambda\varphi \le \psi \le \lambda\varphi$ for some $\lambda \ge 0$, supported with the norm

$$\|\psi\|^\varphi = \inf\{\lambda \ge 0\colon -\lambda\varphi \le \psi \le \lambda\varphi\}.$$

Observe that if $-\lambda\varphi \le \psi \le \lambda\varphi$ then $0 \le \frac{1}{2}(\lambda\varphi - \psi) \le \lambda\varphi$, $0 \le \frac{1}{2}(\lambda\varphi + \psi) \le \lambda\varphi$ and $\psi = \frac{1}{2}(\lambda\varphi + \psi) - \frac{1}{2}(\lambda\varphi - \psi)$. Therefore the space Y is generated by its positive part Y^+. One can verify in a standard way that the restriction operation $\Psi \mapsto \Psi|_{\mathcal{M}^{\mathrm{sa}}}$ determines an isometric and order isomorphism between the Banach conjugate space $(L_1(\varphi)^{\mathrm{sa}})^*$ and Y; and we will identify these spaces.

Associate with the form a the linear functional F_a on Y in the following way.

a) If $0 \le \psi \le \lambda\varphi$ and $\psi = \sum_{i\in I} \langle \cdot g_i, g_i\rangle$ then $g_i \in D_\varphi$, and we set

$$F_a(\psi) \equiv \sum_{i\in I} a(g_i, g_i).$$

The value $F_a(\psi)$ is defined correctly. Indeed, let $\psi = \sum_{j\in J} \langle \cdot h_j, h_j\rangle$ be another representation of ψ. Then, assuming that $\lambda = 1$ for laying out simplification, we have

$$\varphi = \sum_{i\in I} \langle \cdot g_i, g_i\rangle + \sum_{k\in K} \langle \cdot l_k, l_k\rangle = \sum_{j\in J} \langle \cdot h_j, h_j\rangle + \sum_{k\in K} \langle \cdot l_k, l_k\rangle$$

for some $l_k \in H$. Consequently,

$$\sum_{i\in I} a(g_i, g_i) + \sum_{k\in K} a(l_k, l_k) = \sum_{j\in J} a(h_j, h_j) + \sum_{k\in K} a(l_k, l_k),$$

hence, $\sum_{i\in I} a(g_i, g_i) = \sum_{j\in J} a(h_j, h_j)$.

b) The functional F_a defined above on Y^+ is additive and positively homogeneous, therefore it can be uniquely extended to the linear functional on Y.

It is easily seen that F_a has the property:

$$\textit{if}\ \ \psi, \psi_n \in Y^+\ \ \textit{and}\ \ \psi = \sum_{n=1}^{\infty} \psi_n\ \ \textit{then}\ \ F_a(\psi) = \sum_{n=1}^{\infty} F_a(\psi_n). \tag{3}$$

It follows, in particular, that F_a is bounded. Indeed, it suffices to prove that

$$\sup\{|F_a(\psi)|\colon 0 \le \psi \le \varphi\} < \infty.$$

If the latter were false, there would exist a sequence (ψ_n) such that $0 \le \psi_n \le \varphi$ and $|F_a(\psi_n)| \ge 2^n$. Consider $\psi = \sum\limits_{n=1}^{\infty} \dfrac{\psi_n}{2^n}$. Then $0 \le \psi \le \varphi$, while the series $\sum\limits_{n=1}^{\infty} F_a\big(\dfrac{\psi_n}{2^n}\big)$ does not converge, a contradiction.

Thus, $F_a \in Y^*$.

Now, consider the mapping γ which is the isometric and order isomorphism of $L_1(\varphi)^{\mathrm{sa}}$ onto $\mathcal{M}_*^{\mathrm{sa}}$ (see [9, Theorem 17.1 and Theorem 17.6], [7, Theorem 2]). Then γ^* is the isometric and order isomorphism of $(\mathcal{M}_*^{\mathrm{sa}})^* = \mathcal{M}^{\mathrm{sa}}$ onto $(L_1(\varphi)^{\mathrm{sa}})^* = Y$ and γ^{**} is the isometric and order isomorphism of Y^* onto $(\mathcal{M}^{\mathrm{sa}})^*$.

Let us show that the functional $\gamma^{**}(F_a)$ on $\mathcal{M}^{\mathrm{sa}}$ is σ-weakly continuous. Take x_n, x in $\mathcal{M}^+$ such that $x = \sum\limits_{n=1}^{\infty} x_n$ in the sense of σ-weak topology on $\mathcal{M}^{\mathrm{sa}}$, that is equivalent to $x = \sup\limits_k \sum\limits_{n=1}^{k} x_n$. Then $\gamma^*(x) = \sum\limits_{n=1}^{\infty} \gamma^*(x_n)$ and we have by (3):

$$\gamma^{**}(F_a)(x) = F_a(\gamma^*(x)) = \sum_{n=1}^{\infty} F_a(\gamma^*(x_n)) = \sum_{n=1}^{\infty} \gamma^{**}(F_a)(x_n).$$

It follows (cf. [5, Corollary III.3.11]) that $\gamma^{**}(F_a)$ is σ-weakly continuous, i.e. belongs to $\mathcal{M}_*^{\mathrm{sa}}$. Therefore we can consider the integrable sesquilinear form $\gamma^{-1}(\gamma^{**}(F_a))$ which coincides with a by uniqueness arguments. □

Remark. In the general case of infinite weight the validity of the implication (ii) $\Longrightarrow$ (i) question remains open. However, it follows from results of [1] that the implication holds in the special case of standard trace on the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H (see also [9, Theorem 5.2]).

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